On the possible Methods of Dividing the Net Profits of a Mutual Life Assurance Company amongst the Members. *By* Henry Wilbraham, *Esq., Fellow of Trinity College, Cambridge.*

THE object of this paper is to give a brief exposition of the different possible methods or classes of methods of equitably dividing the net profits of a Mutual Life Assurance Office among the members. By an equitable division must be understood one which is fair not only between persons of different ages entering the Office at the same time, but also between persons entering the Office at different epochs of its existence,—one from which the members of different ages will, relatively to one another, receive the same benefit from the profits during the earlier and growing years of the Office, and during years when the amount of its business is declining, as during its stationary period. I assume the premiums actually paid to be composed of the net premium calculated from a correct life table, and of a charge of 25 per cent. on the net premium added to guard against unfavourable contingencies, and that this additional charge is really the source of all the profit divisible among the members; or at least, that whatever profits arise from any other source may be fairly apportioned in the same proportions as appear equitable on the supposition that the charge is the sole source of profits. The addition to the premiums for defraying the expenses of the Office, for the sake of simplicity, I leave out of account. In consideration of the payments of the successive net premiums, the assured is guaranteed a sum at death such that the present prospective value of that sum is equal to the present value of the successive net premiums. Similarly, in consideration of the successive charges on the premiums, he should have the prospect of receiving a sum or sums either at death or at any times before that; not guaranteed, it is true, to any fixed amount, but variable with the circumstances of the Society, such that the present value of such sums or bonuses shall be equal to the present value of the successive charges. That this equation between the present values ought to subsist is as clear in the case of the charges and bonuses as in that of the net premiums and sums assured. If the system of division of profits be such that this equation always subsist, it must be fair as respects each member; and any Office adopting a system in which it does not subsist is committing as great an injustice towards its members as one which calculates the premiums on false life tables.

The systems of division of profits which may be adopted consistently with this equation may be divided into three classes, all of which are equally equitable, and which are adapted for different classes of persons insuring their lives. The first class is where the amounts of the bonuses actually received will be proportional to the amounts of the charges actually paid (accumulated at compound interest if the bonuses be not paid at intervals of a year only), whatever be the duration of life of the assured. The second class is where the bonus to each person is given at his death, and is, as nearly as the circumstances of the Office will allow, the same whatever be the subsequent duration of life after entrance. The third is where the successive bonuses increase in amount as the time during which the assured has been a member increases; or generally, where the benefit from the bonuses increases with the length of survivorship of the assured in a greater ratio than the charges accumulated at compound interest. Of the first class, the simplest form is when there is an annual audit and the profits are distributed among all who have paid premiums during the past year in proportion to the charge of their premiums. This may be distributed—to those who may die before paying their next premium, as an addition to the sums assured; and to those who survive, in the form of a deduction from their next premium. The bonus will, on an average, be equal to the charge of the premium with a year's interest on it. The duty of the charges of one year's premiums, in an Office distributing profits on this system, is simply to guarantee the solvency of the Office during that year. This duty being fulfilled, they are at the end of the year returned to the members who paid them, increased or diminished as the pecuniary circumstances of the Office permit or require, and the solvency of the Office during the next year is guaranteed by the next charges of premiums received. Thus a person insuring will have to pay the whole charged premium the first year; and every subsequent year a premium not far different, either above or below, from the net premium, but in no case exceeding the nominal or charged premium. This is probably the most convenient form of division of profits to persons making an assurance on the life of a debtor as part of an arrangement for securing the sum due to his creditor, and in other cases where a fixed sum and no more is to be secured at as small a rate as possible. A modification of this plan is where the sum apportioned to each person as profit each year is altered into an addition to the sum assured to him. In this case the addition to the assurance of each person must be such as

the amount due to him by way of bonus will, considered as a single premium, insure to him at death; *i.e.*, the amount due as bonus multiplied into $\frac{D^m}{M_m}$, or rather into $\frac{4}{5}\frac{D_m}{M_m}$, as the amount now due should be considered as the *charged* premium of the sum credited to the member at death. If the additional sum assured be calculated on the bonus now due as the single *charged* premium, then in strictness the member ought in subsequent years to receive bonuses on the charges of the annual premiums equivalent to such single premium, as well as on the charges of the annual premiums which he actually pays. If the computation and distribution of profits be not annual but at intervals of several years, in which case of course only the survivors at the time of distribution will receive the bonus, the plan of distribution must be somewhat altered. The fair way in this case will be to distribute the bonus not exactly in proportion to the charge on the premiums received since the last distribution, but in proportion to the charge on the premium for one year, multiplied by $\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}$ where x is the age of the person at the last division of profits, or at his entrance if he entered subsequently thereto, and $x+n$ his present age.

That this is equitable appears from comparing the present value, at the entrance of a member, of his expectant bonuses, with the present value of the charges on his premiums. The present value of the charge, at 25 per cent. on the net premiums, is $\frac{1}{4}\frac{N_{n-1}}{D_n}$: that of the bonuses, supposing that he enters s years before the next apportionment of profits, and that t years is the interval between two successive apportionments, is

$$\frac{1}{4}\frac{N_{n-1}-N_{n+s-1}}{D_{n+s}}\frac{l_{n+s}}{l_n}r^s+\frac{1}{4}\frac{N_{n+s-1}-N_{n+s+t-1}}{D_{n+s+t}}\frac{l_{n+s+t}}{l_n}r^{s+t}+\dots.$$

$$=\frac{1}{4}\frac{N_{n-1}-N_{n+s-1}}{D_n}+\frac{1}{4}\frac{N_{n+s-1}-N_{n+s+t-1}}{D_n}+\frac{1}{4}\frac{N_{n+s+t-1}-N_{n+s+rt-1}}{D_n}+\dots$$

$$=\frac{1}{4}\frac{N_{n-1}}{D_n}.$$

Even if the division of profits be annual, this method ought in strictness to be used if those who die in the year are excluded from the benefit of the division. In this case the formula $\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}$ becomes $\frac{D_x}{D_{x+1}}$. This scheme of division is given by Mr. Farr in

his essay published in the Registrar-General's Twelfth Report, and is considered by him the soundest in principle. That the principle of it is sound, I admit; but there are many, or indeed an infinity, of other schemes equally sound.

There is another possible mode of apportionment of profits belonging to this class which may be mentioned, viz., that at the death of each member there should be paid, besides the sum assured, the sum to which the charges of his premiums have amounted, accumulated from the times of their respective payments, at compound interest, with such increase or diminution as the circumstances of the Office allow. This share of the profits being assigned only at the death of each member, and not paid or credited to him during his lifetime, there is no annual or periodical division of profits. Hence there will, if the business of the Office be successful, be continually a large surplus fund available for bonuses, besides the stock required to balance the present value of the liabilities of the Office. There ought, in fact, to be at any time a surplus equal to the amount of the charges of premiums of each present member of the Society, from his entrance to the present time, accumulated at compound interest. The sum, then, which a member dying will receive from the surplus fund, must be to the accumulated charges on his premiums as the actual surplus is to the surplus which ought to exist—that is, to the total amount of charges of every member accumulated. It might be calculated every year what the surplus actually is, and also what according to this rule it ought to be, and the ratio of the former to the latter expressed as a decimal. Every person dying within the next year will receive, besides the sum originally assured to him, the amount of his charges accumulated at compound interest, multiplied by this decimal. A method which might possibly at first sight appear equitable would be, to divide the whole of each year's profits among those dying in that year in proportion to the amounts of their accumulated charges, instead of paying these additions out of a permanent surplus fund. This would be equitable as between persons of different ages when the Office had reached, and so long as it continued in, a stationary condition, when the net premiums during each year are on an average equal to the assurances payable during that year, and the charges in each year to the amount of accumulated charges of the persons dying in that year; but in any other condition of the business it would not be so. At the commencement of the business of an Office, few would probably die the first year, but many more during any year ten or twelve years thence;

so those dying in the first year would probably receive more than those dying ten or twelve years later, though the latter would have paid much more in charges. It would therefore be far too advantageous at the commencement for the older lives.

The next class is where the bonus is given at the death of each member, and is, as nearly as the circumstances of the Office will permit, the same in amount whatever length of time the member live after entrance; assimilating as nearly as possible the form in which the members receive the equivalent for the charges paid by them with the form in which they receive the equivalent for their net premiums. In this case, as in the last mentioned, there ought to be continually a large surplus fund available for bonuses, which ought never during the continuance of the business to be exhausted, and which clearly ought at any time to be to the ordinary stock of the Society as the charge is to the net premium; *i. e.*, as $\frac{1}{4}$ to 1. The plan of apportionment to each person dying will be very simple. Each year it should be calculated—1st, what ought to be the stock in hand from the net premiums only; 2ndly, by how much the money actually in hand exceeds this; the ratio of the latter to the former should be expressed in the form of a decimal, and any persons dying within the year will, besides the sum assured to them, receive a sum equal to the sum assured multiplied by this decimal.

It is the same, to a person insuring his life, as if an assurance were granted calculated on the whole premium paid as if it were a *net* premium, excepting that only four fifths of the sum are absolutely guaranteed, the remaining one fifth being subject to variation. It might be proposed to divide every year the profit of that year among the persons dying in the same year, in the ratio of the sums assured to them; but this plan would be open to the same objections as the plan before alluded to for dividing every year the yearly surplus to the members dying in the ratio of their accumulated charges. It needs no proof, in this case, that the equation between the present value at entrance of the future charges, and future bonus, is satisfied.

The third class is susceptible of infinite variety. It embraces those systems in which the bonus, if distributed periodically, is not uniform, but to each member increases in amount as the time during which he has been a member increases; or, if apportioned only at death, begins by being less than the accumulated charges if the member die soon after entrance, but becomes larger if he survive long. The correct calculation on any such system must

generally be a work of much labour. The following is an instance of this class. Suppose that the assured does not become entitled to participate in the profits till s years after entrance, and then receives a bonus which increases every subsequent year in arithmetical progression by its first amount multiplied by k. It is necessary here to make use of the equation between the present values of the charges and bonuses, to determine what for any age of the member at entrance ought to be the amount of his first bonus. Suppose his annual net premium to be £1, and the charge £$\frac{1}{4}$, n to be his age at entrance, and y_n his first bonus.

$$\frac{1}{4}\frac{N_{n-1}}{D_n}=y_n\frac{l_{n+s}}{l_n}r^s+y_n(1+k)\frac{l_{n+s+1}}{l_n}r^{s+1}+y_n(1+2k)\frac{l_{n+s+2}}{l_n}r^{s+2}+\,..$$

$$=y_n\frac{D_{n+s}+D_{n+s+1}+\,..}{D_n}+ky_n\frac{D_{n+s+1}+2D_{n+s+2}+\,..}{D_n}$$

$$=y_n\frac{N_{n+s-1}}{D_n}+ky_n\frac{S_{n+s}}{D_n}$$

$$\therefore\ y_n=\frac{1}{4}\frac{N_{n-1}}{N_{n+s-1}+kS_{n+s}}.$$

If the commencement of participation be deferred n years, but then the amount of bonus each year remain unchanged, k will be reduced to o, and $y_n=\frac{1}{4}\frac{N_{n-1}}{N_{n+s-1}}$. Whatever be the fixed value given in the Office to s and k, y may be thus found and tabulated for every age of life; and y_n, $y_n(1+k)$, $y_n(1+2k)$, &c., will be the successive bonuses which a person entering at the age of n years ought to receive if the Office has average success. What he will actually receive in the several years must be to y_n, $y_n(1+k)$, &c., as the total surplus fund actually available for bonuses is to the sum which ought to be the surplus. The latter must be the difference between the present value of all prospective bonuses to all present members, and the present value of all prospective charges on their premiums. Suppose a person to have entered at the age of n years, and to have been g years a member. If g be not less than s—*i. e.*, if he has already come into participation, the present value of his bonuses, including the one immediately payable, will be

$$y_n\left\{1+(g-s)k\right\}+y_n\left\{1+(g-s+1)k\right\}\frac{l_{n+g+1}}{l_{n+g}}r+y_n\left\{1+(g-s+2)k\right\}\frac{l_{n+g+2}}{l_{n+g}}r^2+\ldots$$

$$=y_n\{1+(g-s)k\}\left\{1+\frac{D_{n+g+1}}{D_{n+g}}+\frac{D_{n+g+2}}{D_{n+g}}+\ldots\right\}+ky_n$$
$$\left\{\frac{D_{n+g+1}+2D_{n+g+2}+\ldots}{D_{n+g}}\right\}$$
$$=y_n\{1+(g-s)k\}\frac{N_{n+g-1}}{D_{n+g}}+ky_n\frac{S_{n+g}}{D_{n+g}}.$$

If g be less than s, the present value will be

$$y_n\frac{l_{n+s}}{l_{n+g}}r^{s-g}+y_n(1+k)\frac{l_{n+s+1}}{l_{n+g}}+\ldots$$
$$=y_n\frac{D_{n+s}+(1+k)D_{n+s+1}+\ldots}{D_{n+g}}$$
$$=y_n\frac{N_{n+s-1}+kS_{n+s}}{D_{n+g}}=\frac{1}{4}\frac{N_{n-1}}{D_{n+g}}.$$

The present value of future charges, including the one presently payable, is $\frac{1}{4}\frac{N_{n+g-1}}{D_{n+g}}$. Therefore the present value of future bonuses, minus that of future charges, for a person aged $n+g$ who entered at the age of n, will be $z_{n,g}=\frac{N_{n-1}-N_{n+g-1}}{4D_{n+g}}$, or $y_n\{1+(g-s)k\}\frac{N_{n+}{}^{g-1}}{D_{n+g}}+ky_n\frac{S_{n+g}}{D_{n+g}}-\frac{1}{4}\frac{N_{n+g-1}}{D_{n+g}}$, according as g is or is not less than s. If the bonuses, when once begun, remain constant, and therefore $k=0$, the latter formula becomes $\frac{N_{n+g-1}}{N_{n+s-1}}$ $\frac{N_{n-1}-N_{n+s-1}}{4D_{n+g}}$. If an Office were to divide its profits on this system, the successive values of y_n for every age, and also the values of $z_{n,g}$ for all values of n and g, must be tabulated. That being once done, the computation of the shares of the members in the profits would be very simple. Each year the actual surplus would be computed; also the premium of each member, multiplied by $z_{n,g}$ according to his age and standing in the Society: the latter must be divided by the former and expressed as a decimal. Each person would receive y_n, $y_n(1+k)$ . . . according to the time he has been in the Society, multiplied by this decimal.

Another scheme of this class would be, if the member, as in the last case, comes into participation after s years, and then an addition y is made to the sum assured to him, next year an addition $(1+k)y$, and so on, the additions increasing in arithmetic progression. To determine y, supposing as before £1 to be the net premium, and n the age at entrance, we have the equation

$$\tfrac{1}{4}\frac{N_{n-1}}{D_n}=y_n\frac{M_n}{D_n}\frac{l_{n+s}}{l_n}+y_n(1+k)\frac{M_n}{D_n}\frac{l_{n+s+1}}{l_n}+\ldots$$

$$=y_n\frac{M_n}{D_n}\left\{\frac{l_{n+s}+l_{n+s+1}+\ldots}{l_n}+k\frac{l_{n+s+1}+2l_{n+s+2}+\ldots}{l_n}\right\}$$

$$y_n=\frac{1}{4}\frac{N_{n-1}}{M_n}\cdot\frac{l_n}{l_{n+s}+l_{n+s+1}+\ldots+k\{l_{n+s+1}+2l_{n+s+2}+\ldots\}}$$

$$=\tfrac{1}{4}\frac{l_n}{l_{n+s}+l_{n+s+1}+\ldots+k\{l_{n+s+1}+2l_{n+s+2}+\ldots\}}\times\text{sum assured.}$$

The present value of the future additions hereafter to be credited to a person who has been g years a member will be, if g be less than s,

$$y\frac{M_{n+g}}{D_{n+g}}\left\{\frac{l_{n+s}}{l_{n+g}}+\frac{l_{n+s+1}}{l_{n+g}}(1+k)+\ldots\right\}$$

$$=\tfrac{1}{4}\frac{M_{n+g}}{D_{n+g}}\cdot\frac{l_n}{l_{n+g}}\times\text{sum assured;}$$

or, if g be not less than s,

$$y_n\frac{M_{n+g}}{D_{n+g}}\left\{1+(g-s)k+\frac{\{1+(g-s+1)k\}\,l_{n+g+1}}{l_{n+g}}+\ldots\right\}$$

$$=\tfrac{1}{4}\frac{M_{n+g}}{D_{n+g}}\frac{l_n}{l_{n+g}}\cdot\frac{N_{n-1}}{M_n}\,\frac{(1+\overline{g-s}\,k)\{l_{n+g}+l_{n+g+1}+\ldots\}+k\{l_{n+g+1}+2l_{n+g+2}+\ldots\}}{l_{n+s}+l_{n+s+1}+\ldots+k\{l_{n+s+1}+2l_{n+s+1}+\ldots\}};$$

and $z_{n,g}$ is equal to this present value, minus $\tfrac{1}{4}\frac{N_{n+g-1}}{D_{n+g}}$. As before, the values of y and z must be calculated and tabulated, and every year the actual surplus must be computed, and the sum of the premiums of each member multiplied by $z_{n,g}$, and the ratio of the former to the latter found as a decimal; the sum to be added to each assurance will be $y, y(1+k)\ldots$ multiplied by this number.

If an Office makes the additions proportional simply to the sums assured, and increasing every year in arithmetical progression, the ratio of the increase of the sum assured to a person entering at the age of n, to one entering at the age of $n+q$, will be $\frac{\text{sum assured at age } n}{\text{sum assured at age } n+q}$; whereas it ought to be—

$$\frac{\text{Sum assured at age } n}{\text{Sum assured at age } n+q}\times\frac{\dfrac{l_{n+q+s}+l_{n+q+s+1}+\ldots}{l_{n+q}}+k\dfrac{l_{n+q+s+1}+2l_{n+q+s+2}+\ldots}{l_{n+q}}}{\dfrac{l_{n+s}+l_{n+s+1}+\ldots}{l_n}+k\dfrac{l_{n+s+1}+2l_{n+s+2}+\ldots}{l_n}}$$

which will generally be considerably smaller. Such a scheme, therefore, is more favourable to young lives than it ought to be.

Supposing now a division of profits to be made once every s years, then, for a person who entered immediately after one division of profits, we should have

$$y_n = \tfrac{1}{4}\frac{N_{n-1}}{M_n}\cdot\frac{l_n}{l_{n+s}+l_{n+2s}+\ldots+k(l_{n+2s}+2l_{n+3s}+\ldots)}$$

$$=\tfrac{1}{4}\frac{l_n}{l_{n+s}+l_{n+2s}+\ldots+k(l_{n+2s}+2l_{n+3s}+\ldots)}\times \text{sum assured by an annual premium of £1.}$$

If the time of his entrance be a years after the last division,

$$y_n = \tfrac{1}{4}\frac{N_{n-1}}{M_n}\,\frac{l_n}{l_{n-a+s}+l_{n-a+2s}+\ldots+k(l_{n-a+2s}+2l_{n-a+3s}+\ldots)}$$

$$=\tfrac{1}{4}\frac{l_n}{l_{n-a+s}+\ldots+k(l_{n-a+2s}+2l_{n-a+3s}+\ldots)}\times \text{sum assured.}$$

For this system of division the quantity $l_{n+s}+l_{n+2s}+\ldots+k(l_{n+2s}+\ldots)$ should be tabulated for every value of n, s being constant. If this be called F_n, then, if the member enter immediately after a division of profits, $y_n=\frac{1}{4}\frac{l_n}{F_n}\times$ sum assured. If he enter a years after the last division, $y_n=\frac{1}{4}\frac{l_n}{F_{n-a}}\times$ sum assured.

To find the surplus which ought to exist for payment of bonuses, we have, for a person now aged $n+g$ years who entered the Society at the age of n, and who has already had additions to his assurance made to him at p different times of division of profits, the present value of the future additions to his assurance, including the one to be now made:—

$$y_n\frac{M_{n+g}}{D_{n+g}}\left\{(1+pk)+(1+\overline{p+1}k)\frac{l_{n+g+s}}{l_{n+g}}+(1+\overline{p+2}k)\frac{l_{n+g+2s}}{l_{n+g}}+\ldots\right\}$$

$$=y_n\frac{M_{n+g}}{D_{n+g}}\,\frac{1}{l_{n+g}}\left\{F_{n+g}+pkG_{n+g}\right\},\text{ where } G_n=l_n+l_{n+s}+l_{n+2s}+\ldots$$

$$z_{n,g}=y_n\frac{M_{n+g}}{D_{n+g}}\cdot\frac{1}{l_{n+g}}\left\{F_{n+g}+pkG_{n+g}\right\}-\tfrac{1}{4}\frac{N_{n+g-1}}{D_{n+g}}\,.$$

From these values of y and z, the sums to be added to each person's assurance must be calculated as before.

It will be observed, that in any of these methods of division of profits in which the bonus is given in the form of an addition to the sum assured, except the second class, the apportionment may

take place either to all members at fixed intervals of time, annually or otherwise, or only to each member at his death. The difference will not be great as regards any member. If, for instance, to each member a sum is annually credited proportional to the charge on premium paid by him in that year, but is suffered to accumulate in the Office till his death, at compound interest, it is much the same to him as if no sum were credited to him before his death, but that then a sum were given him (additional to the sum assured) proportional to the charges paid ever since he became a member, accumulated at compound interest: the difference of the two amounts would probably be very trifling. But as respects the security of the Office, the latter plan would be much more advantageous. In the former plan, the only security against unfavourable contingencies would be the charges on the premiums for a single year; in the latter, the security would be the whole accumulated amount of the charges of each present member, from the time of his entrance to the present time. And it will be generally the case, that the systems of division of profits in which the apportionment is made only at the deaths of the members afford greater security against casualties to the Office than those in which the division is periodically made, on whatever system it be made, and whether by way of bonus presently payable or of addition to the sum assured; or at least that a greater percentage ought in the latter case to be added to the net premium as a charge in order to give equal security.

On the Practical Application of the Doctrine of Chances, as it regards the Subdivision of Risks. By the late Dr. Thomas Young.

[Extracted from *The Quarterly Journal of Science.*]

It is well known to those who have studied the theory of chances, that where the magnitude of a risk may be divided into an indefinite number of parts, it is possible, by this multiplication of smaller parts, to confine the probability of the occurrence of any given excess or deficiency in the result, above or below its mean value, within any given limits; and it is of great practical importance to ascertain both how far the subdivision ought to be carried with regard to insurances, and what are the inconveniences to be apprehended from the admission of occasional deviations from the general rule.

We may take, for example, the case of an Office having under-